J. Math. Anal. Appl. 351 (2009) 595–602

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mixed initial boundary value problem for hyperbolic geometric flow on Riemann surfaces ☆

Fagui Liu

North China Institute of Water Conservancy and Hydroelectric Power, Zhengzhou 450011, PR China

ARTICLE INFO

Article history:
Received 31 January 2008
Available online 29 October 2008
Submitted by H.R. Parks

Keywords:
Hyperbolic geometric flow
Mixed initial boundary value problem
Riemann surface
Classical solution

ABSTRACT

The hyperbolic geometric flow equations is introduced recently by Kong and Liu motivated by Einstein equation and Hamilton Ricci flow. In this paper, we consider the mixed initial boundary value problem for hyperbolic geometric flow, and prove the global existence of classical solutions. The results show that, for any given initial metric on R^2 in certain class of metric, one can always choose suitable initial velocity symmetric tensor such that the solutions exist, and the scalar curvature corresponding to the solution metric g_{ij} keeps bounded. If the initial velocity tensor does not satisfy the certain conditions, the solutions will blow up at a finite time. Some special explicit solutions to the reduced equation are given.

1. Introduction

Let $\mathcal{M}$ be an n-dimensional complete Riemann manifold with Riemann metric g_{ij}, the following general evolution equation for the metric g_{ij}

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathcal{F}\left(g, \frac{\partial g}{\partial t}\right) = 0$$

has been recently introduced by Kong and Liu [1] and named as general *version of hyperbolic geometric flow* (denoted by HGF), where $\mathcal{F}$ are some given smooth functions of the Riemannian metric g and its first derivative with respect to t, R_{ij} are Ricci curvature tensor. In the present paper, HGF is named as Kong-Liu's hyperbolic geometric flow. We can see the celebrated survey paper [2] for the progress on this new topic.

In this paper, we study the evolution of a Riemannian metric g_{ij} on a Riemann surface $\mathcal{M}$ by its Ricci curvature tensor R_{ij} under the HGF equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}, \tag{1}$$

and the Ricci curvature given by

$$R_{ij} = \frac{1}{2} R g_{ij}. \tag{2}$$

☆ Supported by the NSF of China (10571024) and the Basic Research of Henan Province.

E-mail address: liufagui@ncwu.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.10.043

Thus, the HGF equation simplifies the following equation for the special metric

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -R g_{ij}, \tag{3}$$

where the scalar curvature function $R = 2K$ in which K is the Gauss curvature.

Here, we are interested in the following initial metric on the surface of topological type R^2

$$t = 0: \quad ds^2 = u_0(x)\big(dx^2 + dy^2\big), \tag{4}$$

where $u_0(x) \in C^2$ with bounded C^2 norm and satisfies

$$0 < k \leqslant u_0(x) \leqslant M < \infty, \tag{5}$$

in which k and M are positive constants.

The metric for a surface can always be written (at least locally) as

$$g_{ij} = u(x, y, t)\delta_{ij}, \tag{6}$$

where $u(t, x, y) > 0$, and δ_{ij} is the Kronecker's symbol. Therefore, we have

$$R = -\frac{\Delta \ln u}{u}. \tag{7}$$

Thus, Eq. (3) can be deduced to

$$u_{tt} - \Delta \ln u = 0. \tag{8}$$

Note that the initial data $u_0(x)$ only depend on x and is independent of y, Kong and Liu [3,4] have considered the following special model

$$u_{tt} - (\ln u)_{xx} = 0, \tag{9}$$

where $u = u(x, t) > 0$, and proved the global existence of classical solution and blow-up phenomena for Cauchy problem of (9).

In this paper, we will consider the mixed initial boundary value (denoted by MIBV) problem of (9) with the following data:

$$t = 0: \quad u = u_0(x), \quad u_t = u_1(x), \quad x \geqslant 0, \tag{10}$$

$$x = 0: \quad u_x = 0, \quad t \geqslant 0, \tag{11}$$

where $u_1(x) \in C^1(R^+)$ with bounded C^1 norm.

Theorem 1. *Suppose that the conditions of C^2 compatibility are satisfied at point $(0, 0)$, and*

$$u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}} \geqslant 0, \quad \forall x \in R^+, \tag{12}$$

$$u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}} \geqslant 0, \quad \forall x \in R^+. \tag{13}$$

Then, the MIBV problem (9)–(11) admits a unique global solution on domain

$$D = \big\{(x, t) \bigm| x \geqslant 0,\ t \geqslant 0\big\}.$$

Remark 1. Following Theorem 1, it is easy to see that the MIBV problem

$$\begin{cases} \frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} \quad (i, j = 1, 2), \\ t = 0: \quad g_{ij} = u_0(x)\delta_{ij}, \quad \frac{\partial g_{ij}}{\partial t} = u_1(x)\delta_{ij} \quad (i, j = 1, 2),\ x \geqslant 0, \\ x = 0: \quad \frac{\partial g_{ij}}{\partial x} = 0 \end{cases} \tag{14}$$

has a unique smooth solution on D, and the solution metric g_{ij} admits the following form

$$g_{ij} = u(x, t)\delta_{ij} \quad (i, j = 1, 2). \tag{15}$$

Remark 2. Let $u_0(x) \equiv 1$, $u_1(x) \equiv C$ (C is a constant). Then the MIBV (9)–(11) admits exact solution $u(x, t) \equiv Ct + 1$.

Theorem 2. *Suppose that the conditions of C^2 compatibility are satisfied at point* $(0,0)$, (13) *holds and there exists a point* $x_0 \in R^+$ *such that*

$$u_1(x_0) + \frac{u_0'(x_0)}{\sqrt{u_0(x_0)}} < 0. \tag{16}$$

Then, the classical solutions to the MIBV problem (9)–(11) *must blow up in the finite time.*

Theorems 1 and 2 give a global existence and nonexistence result on the classical solutions of hyperbolic geometric flow. They show that if we choose suitable initial velocity tensor $\frac{\partial g_{ij}(x,0)}{\partial t}$, the solution to the MIBV problem admits a unique classical solutions on the domain D. Otherwise, the solutions to the MIBV problem will blow up in the finite time.

The paper is organized as follows: some preliminaries is derived in Section 2, Theorem 1 and Theorem 2 are proved in Section 3 and Section 4, respectively. Some special explicit solutions to the reduced equation are given in Section 5.

2. Preliminaries

Let

$$u_t = v, \qquad u_x = w, \tag{17}$$

then, it follows from (9) that

$$\begin{cases} u_t = v, \\ w_t - v_x = 0, \\ v_t - \dfrac{1}{u} w_x = -\dfrac{w^2}{u^2}. \end{cases} \tag{18}$$

It is easy to see that the eigenvalues of Eqs. (18) are

$$\lambda_1 = -\lambda, \qquad \lambda_2 = 0, \qquad \lambda_3 = \lambda, \qquad \lambda = \frac{1}{\sqrt{u}}, \tag{19}$$

the matrix $L(U)$ $(U = (u, w, v)^T)$ of left eigenvectors and the matrix $R(U)$ of right eigenvectors are, respectively,

$$L(U) = \begin{pmatrix} l_1(U) \\ l_2(U) \\ l_3(U) \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & -\lambda & 1 \end{pmatrix}, \qquad R(U) = \big(r_1(U), r_2(U), r_3(U)\big) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & 0 & -\lambda \\ 1 & 0 & 1 \end{pmatrix}.$$

Since

$$\nabla \lambda_i(U) r_i(U) \equiv 0 \quad (i = 1, 2, 3),$$

so that Eqs. (18) is a linearly degenerate strict hyperbolic system.

Set

$$m = \sqrt{\lambda}(v + \lambda w), \qquad n = \sqrt{\lambda}(v - \lambda w). \tag{20}$$

Lemma 2.1. *m and n satisfy*

$$\begin{cases} m_t - \lambda m_x = -\dfrac{1}{4}\lambda^{\frac{2}{3}} m^2, \\ u_t = \dfrac{1}{2\sqrt{\lambda}}(m + n), \\ n_t + \lambda n_x = -\dfrac{1}{4}\lambda^{\frac{3}{2}} n^2. \end{cases} \tag{21}$$

Proof. Noting that

$$\lambda_t = -\frac{1}{2}\lambda^3 v, \qquad \lambda_x = -\frac{1}{2}\lambda^3 w,$$

we can calculate that

$$\begin{aligned} p_t - \lambda p_x &= (v + \lambda w)_t - \lambda(v + \lambda w)_x = v_t - \lambda v_x + \lambda(w_t - \lambda w_x) + w(\lambda_t - \lambda \lambda_x) = -\frac{1}{2}\lambda^3 w(v + \lambda w) \\ &= \frac{\lambda^2}{4}(q - p)p, \end{aligned} \tag{22}$$

$$q_t + \lambda q_x = \frac{\lambda^2}{4}(p - q)q, \tag{23}$$

where

$$p = v + \lambda w, \qquad q = v - \lambda w. \tag{24}$$

Noting that

$$\frac{\lambda^2}{4} q = \frac{1}{4}\big[(\ln u)_t - \lambda(\ln u)_x\big], \qquad \frac{\lambda^2}{4} p = \frac{1}{4}\big[(\ln u)_t + \lambda(\ln u)_x\big],$$

we can easy get the first equation in (21).

In a similar way, we can prove the third equation in (21). The second equation in (21) is obvious.

Thus, the proof is complete. □

Let

$$r = p_x, \qquad s = q_x, \tag{25}$$

by a direct calculation, we easily give the following lemma.

Lemma 2.2. *r, s, u_x and u_{xx} satisfy*

$$r_t - \lambda r_x = A_1 r + A_2 s + A_3, \tag{26}$$

$$s_t + \lambda s_x = B_1 s + B_2 r + B_3, \tag{27}$$

$$(u_x)_t = \frac{1}{2}(r + s), \tag{28}$$

$$u_{xx} = \frac{1}{2} u^{\frac{1}{2}} (r - s) + \frac{1}{2u} u_x^2, \tag{29}$$

where

$$A_1 = \frac{\lambda^2(2q - 3p)}{4}, \qquad A_2 = \frac{\lambda^2 p}{4}, \qquad A_3 = \frac{\lambda^3}{8}(p - q)^2 p,$$

and

$$B_1 = \frac{\lambda^2(2p - 3q)}{4}, \qquad B_2 = \frac{\lambda^2 q}{4}, \qquad B_3 = \frac{\lambda^3}{8}(p - q)^2 q.$$

It is easy to find from (11), (17), (19)–(20) and (24)–(25) that

$$t = 0: \quad \begin{cases} m = \dfrac{1}{\sqrt[4]{u_0(x)}}\left(u_1(x) + \dfrac{u_0'(x)}{\sqrt{u_0(x)}}\right), \\[2ex] n = \dfrac{1}{\sqrt[4]{u_0(x)}}\left(u_1(x) - \dfrac{u_0'(x)}{\sqrt{u_0(x)}}\right), \end{cases} \tag{30}$$

$$x = 0: \quad n = m, \tag{31}$$

and

$$t = 0: \quad \begin{cases} r = u_1'(x) - \dfrac{u_0'(x)}{2u_0^{\frac{3}{2}}(x)} + \dfrac{u_0''(x)}{u_0^{\frac{1}{2}}(x)}, \\[2ex] s = u_1'(x) + \dfrac{u_0'(x)}{2u_0^{\frac{3}{2}}(x)} - \dfrac{u_0''(x)}{u_0^{\frac{1}{2}}(x)}, \end{cases} \tag{32}$$

$$x = 0: \quad s = r. \tag{33}$$

3. Global classical solution—proof of Theorem 1

According to the local existence and uniqueness theorems of classical solutions to quasilinear hyperbolic systems [5], in order to prove Theorem 1, it suffices to establish the uniform a priori estimate on the C^1 norm of m, n and C^2 norm of u.

Let $x = x_i(t, \beta_i)$ ($x_i(0, \beta_i) = \beta_i$, $i = 1, 2, 3$) denote λ_i ($i = 1, 2, 3$) characteristics passing through point $(x, t) \in D$ with

$$\frac{dx_i}{dt} = \lambda_i, \quad i = 1, 2, 3,$$

and $x = x_3(t)$ ($x_3(\tau) = 0$) be λ_3 characteristics passing through point (s, t) and intersecting the line $x = 0$ at $(0, \tau)$ ($\tau \leqslant t$). Then, the domain D is divided by D_1 and D_2 with

$$D_1 = \big\{(x, t) \,\big|\, x \geqslant x_3(t, 0),\ t \geqslant 0\big\}, \qquad D_2 = \big\{(x, t) \,\big|\, 0 \leqslant x \leqslant x_3(t, 0),\ t \geqslant 0\big\}. \tag{34}$$

Lemma 3.1. *On the existence domain of the classical solution to the MIBV problem, if* (12)–(13) *hold, then*

$$0 \leqslant m(x,t) \leqslant \sup_{x\in R^+} m_0(x), \quad x \geqslant 0,\ t \geqslant 0, \tag{35}$$

$$0 \leqslant n(x,t) \leqslant \sup_{x\in R^+} n_0(x), \quad x \geqslant 0,\ t \geqslant 0, \tag{36}$$

$$0 < \inf_{x\in R^+} u_0(x) \leqslant u(x,t) \leqslant \sup_{x\in R^+} u_0(x) + Ct, \quad x \geqslant 0,\ t \geqslant 0, \tag{37}$$

where $C > 0$ is a constant.

Proof. On the domain D_1, we can use known results on Cauchy problem [3,4], if (12) and (13) hold, (35)–(37) hold.

On the domain D_2, along the λ_1 characteristics, we have

$$m(x,t) = \frac{m_0(\beta_1)}{F(\beta_1,t)}, \tag{38}$$

where

$$F(\beta_1,t) = 1 + \frac{m_0(\beta_1)}{4}\int_0^t \lambda^{\frac{3}{2}}\big(x_1(\tau,\beta_1),\tau\big)\,d\tau. \tag{39}$$

Noting that $\lambda^{\frac{3}{2}} = u^{-\frac{3}{4}} > 0$, $m_0(x) \geqslant 0$, it is easy to see that (35) holds.

On the domain D_2, along the λ_3 characteristics, it holds that

$$n(x,t) = \frac{n(\tau,0)}{1 + \frac{n(\tau,0)}{4}\int_\tau^t \lambda^{\frac{3}{2}}(x_3(s),s)\,ds}. \tag{40}$$

Since $n(0,\tau) = m(0,\tau)$ and (35), we can get that (36) holds.

By $(21)_2$, we have

$$\big(u^{\frac{3}{4}}\big)_t = \frac{3}{8}(m+n). \tag{41}$$

Integrating (41), we get

$$u^{\frac{3}{4}}(x,t) = u_0^{\frac{3}{4}}(\beta_2) + \frac{3}{8}\int_0^t (m+n)\big(x_2(\tau,\beta_2),\tau\big)\,d\tau. \tag{42}$$

Thus, by (35) and (36), we can obtain (37).

Lemma 3.1 is thus proved. □

Proof of Theorem 1. For any $(x,t) \in D_1$, following [4], there exists a positive constant M_1 such that

$$\big|r(x,t)\big| \leqslant M_1, \qquad \big|s(x,t)\big| \leqslant M_1. \tag{43}$$

For $(x,t) \in D_2$, by (26)–(27), it is easy to see that

$$r(x,t) = r_0(\beta_1) + \int_0^t (A_1 r + A_2 s + A_3)\big(x_1(\tau,\beta_1),\tau\big)\,d\tau, \tag{44}$$

$$s(x,t) = s(0,\tau) + \int_\tau^t (B_1 s + B_2 r + B_3)\big(x_3(s),s\big)\,ds, \quad (x,t) \in D_2. \tag{45}$$

By (33), we have

$$s(0,\tau) = r(0,\tau) = r_0(x_0) + \int_0^\tau (A_1 r + A_2 s + A_3)\big(x_1(s,x_0),s\big)\,ds. \tag{46}$$

Therefore, there exists constant $M(T) > 0$, such that

$$\big|r(x,t)\big| \leqslant M(T), \qquad \big|s(x,t)\big| \leqslant M(T), \tag{47}$$

and

$$\big|u(x,t)\big| \leqslant M(T), \qquad \big|u_x(x,t)\big| \leqslant M(T), \qquad \big|u_{xx}(x,t)\big| \leqslant M(T), \tag{48}$$

where $M(T) > 0$ is a constant.

Thus, by Lemma 3.1, Theorem 1 is proved. □

Remark 3. By (7) and (48), under the assumptions of Theorem 1, there exists a constant $M_1(T) > 0$, such that

$$\left|R(x,t)\right| \leqslant M_1(T).$$

4. Blow-up of classical solutions—proof of Theorem 2

In this section we will investigate the blow-up phenomena of hyperbolic geometric flow.

Proof of Theorem 2. It follows from (21) and Lemma 3.1 that

$$m_t - \lambda m_x \leqslant 0, \qquad n_t + \lambda n_x \leqslant 0.$$

Thus, we have

$$m(x,t) \leqslant \sup_{x \geqslant 0} m_0(x), \quad (x,t) \in D, \tag{49}$$

$$n(x,t) \leqslant \sup_{x \geqslant 0} n_0(x), \quad (x,t) \in D_1, \tag{50}$$

$$n(x,t) \leqslant n(0,\tau), \quad (x,t) \in D_2. \tag{51}$$

By $n(0,\tau) = m(0,\tau)$, we can obtain

$$m(x,t) + n(x,t) \leqslant M_0 \equiv \max\Big\{\sup_{x \geqslant 0} m_0(x), \sup_{x \geqslant 0} m_0(x) + \sup_{x \geqslant 0} n_0(x)\Big\}. \tag{52}$$

Noting that $u_0(x) \geqslant k > 0$, it follows from (14) and (30) that

$$m_0(x_0) < 0. \tag{53}$$

Integrating $(21)_1$, along λ_1 characteristics, we get

$$m(x_0,t) = \frac{m_0(x_0)}{F(t,x_0)}, \tag{54}$$

where

$$F(x_0,t) = 1 + \frac{m_0(x_0)}{4} \int_0^t \lambda^{\frac{3}{2}}\big(x_1(x_0,\tau),\tau\big)\,d\tau, \quad \lambda^{\frac{3}{2}} = u^{-\frac{3}{4}}. \tag{55}$$

By (42) and (52), we have

$$u^{\frac{3}{4}}(x,0) \geqslant k^{\frac{3}{4}}, \qquad u^{\frac{3}{4}}(x,t) \leqslant M^{\frac{3}{4}} + \frac{3}{8} M_0 t. \tag{56}$$

Case (i): If $M_0 < 0$, there exists $\tau_0 = \frac{8M^{\frac{3}{4}}}{3(-M_0)} > 0$, such that

$$u(x,t) \leqslant 0, \quad t \geqslant \tau_0.$$

This implies that the system (9) is meanless as $t \geqslant \tau_0$, that is, the MIVP problem (9)–(11) admits a unique local classical solution.

Case (ii): If $M_0 = 0$, then, it is easy to find that

$$F(x_0,t) \leqslant 1 + \frac{m_0(x_0)}{4} M^{-\frac{3}{4}} t.$$

Noting that $F(x_0,0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_0 = \frac{4M^{\frac{3}{4}}}{-m_0(x_0)} > 0$, such that as $t \to t_0^-$,

$$F(x_0,t) \to 0^+. \tag{57}$$

So that there exists finite time $T = T(x_0) > 0$, such that as $t \to T^-$,

$$m(x_0,t) \to -\infty, \quad t \to T^-. \tag{58}$$

Case (iii): If $M_0 > 0$, then, it follows from (55)–(56) that

$$F(x_0,t) \leqslant 1 + \frac{2m_0(x_0)}{3M_0} \ln\left(1 + \frac{3M_0}{8M^{\frac{3}{4}}} t\right).$$

Thus, noting that $F(x_0,0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_* > 0$, such that as $t \to t_*^-$, (57) holds, and then (58) is obtained.

Theorem 2 is thus proved. □

Remark 4. Noting

$$R(x,t) = -\frac{1}{2}\left[\lambda(r-s) + \frac{1}{4\lambda}(m-n)^2\right],$$

we have

$$R(x,t) \to \infty, \quad t \to t_*^-.$$

5. Explicit solutions

In this section, we will give some special explicit solutions to Eq. (9).

5.1. Solutions to separation variables

Let

$$u(x,t) = f(t)g(x) > 0, \tag{59}$$

then, we have

$$f''(t) - \frac{1}{g(x)}\big(\ln g(x)\big)'' = 0. \tag{60}$$

It follows from (60) that

$$f''(t) = \frac{1}{g(x)}\big(\ln g(x)\big)'' = c, \tag{61}$$

where c is a constant. Thus, we get

$$f(t) = \frac{c}{2}t^2 + c_1 t + c_2, \tag{62}$$

and $g(x)$ satisfies the following equation

$$g(x)g''(x) - g'^2(x) = cg^3, \tag{63}$$

where c_1 and c_2 are constants.

5.2. Solutions to traveling wave type

Let

$$u(x,t) = f(\xi) > 0, \quad \xi = x - at, \tag{64}$$

we have

$$a^2 f''(\xi) f^2(\xi) - f(\xi) f''(\xi) + f'^2(\xi) = 0. \tag{65}$$

Let $p = f'(\xi)$, we have

$$\frac{1}{p}\frac{dp}{dy} = \frac{1}{y(1-a^2 y)}, \quad y = f(\xi). \tag{66}$$

It follows from (66) that

$$p = \frac{dy}{d\xi} = \frac{c_4 y}{1 - a^2 y},$$

where c_4 is a real constant. Therefore, Eq. (9) admits explicit traveling wave solution $u(x,t) = f(x-at)$ which can be solved explicitly in the following implicit from

$$\ln f(x-at) - a^2 f(x-at) = c_4(x-at) + c_5, \tag{67}$$

where c_5 is a real constant.

Remark 5. For Eq. (8), let

$$u(x,y,t) = \phi(\xi) > 0, \quad \xi = Ax + By - at, \tag{68}$$

we get

$$a^2\phi^2(\xi)\phi''(\xi) - \big(A^2 + B^2\big)\phi(\xi)\phi''(\xi) + \big(A^2 + B^2\big)\phi'^2(\xi) = 0. \tag{69}$$

Similarly, we can get Eq. (8) admits explicit traveling wave solution $u(x, y, t) = \phi(Ax + By - at)$ which can be solved explicitly in the following implicit from

$$\ln\phi(Ax + By - at) - \frac{a^2}{A^2 + B^2}\phi(Ax + By - at) = c_6(Ax + By - at) + c_7, \tag{70}$$

where c_6 and c_7 are real constants.

Acknowledgments

The authors are particularly grateful to Professor Kong Dexing for his helpful discussions, and would like to thank the referee for his/her pertinent comments and valuable suggestions.

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